

# An interesting property of the Friedman universes

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## Abstract

We show in the paper that Friedman universes can be created from empty, flat Minkowskian spacetime by using suitable conformal rescaling of the spacetime metric.

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# 1 Friedman universes

Einstein equations and Cosmological Principle lead us together to Friedman universes. These universes give standard mathematical models of the real Universe.

Einstein equations

$$G_{ik} := R_{ik} - \frac{1}{2}g_{ik}R = \frac{8\pi G}{c^4}T_{ik} =: \beta T_{ik} \quad (1)$$

form system of the ten, 2-nd order quasilinear partial differential equations on ten unknown functions. Solving these equations under given initial and boundary conditions one obtains local geometry of the spacetime, i.e.,  $g_{ik}(x) \longrightarrow \Gamma^i_{kl}(x) \longrightarrow R^i_{klm}(x)$  and local distribution and motion of matter, i.e.,  $T_{ik}(x)$ .

Here  $G_{ik}$  is the so-called *Einstein tensor*,  $T_{ik}$  is the *matter energy-momentum tensor* (the source of the gravitational field which is represented by tensor  $G_{ik}$ ),  $c$  is the velocity of light in vacuum, and  $G$  means Newtonian gravitational constant;  $g_{ik}(x)$  denote components of the metric tensor, and  $\Gamma^i_{kl}(x)$ ,  $R^i_{klm}(x)$  are the Levi-Civita connection and Riemannian curvature components respectively.  $R_{ik}$  mean components Ricci tensor and  $R$  is the so-called curvature scalar (See, eg., [1]). All Latin indices take values 0, 1, 2, 3.

The matter tensor  $T_{ik}(x)$  consists of  $g_{ik}$ ,  $u^i$ ,  $p$ ,  $\rho$ , where  $u^i$ ,  $p$ ,  $\rho$  denote 4-velocity, pressure and density of matter respectively.

Cosmological Principle says that in the largest scale the real Universe is homogeneous and isotropic<sup>2</sup>.

In the following we will use *geometrized units* in which  $G = c = 1$ . Friedman universes are cosmological solutions to the Einstein equations constrained by Cosmological Principle and they are foundation of the relativistic cosmology [1, 2].

The line element  $ds^2 = g_{ik}(x)dx^i dx^k$  for these universes, called *Friedman-Lemaitre'-Robertson-Walker* line element, in the *comoving coordinates*  $x^0 = t$ ,  $x^1 = \chi$ ,  $x^2 = \vartheta$ ,  $x^3 = \varphi$ , reads

$$ds^2 = dt^2 - R^2(t)[d\chi^2 + S^2(\chi)(d\vartheta^2 + \sin^2\vartheta d\varphi^2)], \quad (2)$$

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<sup>2</sup>We are modelling *cosmological substrat* by using an ideal (or perfect) fluid.

where

$$\begin{aligned} S(\chi) &= \sin\chi, \text{ if } k = 1 \\ S\chi &= \chi, \text{ if } k = 0 \\ S(\chi) &= \sinh\chi, \text{ if } k = (-)1. \end{aligned} \quad (3)$$

$t$  is the *cosmic time*, i.e., the proper time for *isotropic observers*, which are at rest in the coordinates  $(t, \chi, \vartheta, \varphi)$ .

An isotropic observer  $O$  represents center of mass of a cluster of galaxies in real Universe.  $R(t)$  is the so-called *scale factor* (it scales spatial distances) and  $k = 0, \pm 1$  means the *normalized curvature* (curvature index) of the spatial sections  $x^0 = t = \text{const.}$

If  $k = 1$ , then we have closed (spherical or elliptical) spatial sections, if  $k = 0$  the geometry of the spatial section is flat, and if  $k = (-)1$ , then the geometry of spatial sections is hyperbolic.

Usually one chooses the moment  $t = 0$  of the cosmic time  $t$  when  $R = 0$ , i.e., usually one has  $R(0) = 0$ .

Einstein equations with perfect fluid (incompressible fluid, without any viscosity and not conducting heat) as source <sup>3</sup> reduce, for the FLRW line element (2)-(3) to the *Friedman equations*

$$\frac{3\dot{R}^2}{R^2} + \frac{3k}{R^2} = \frac{\rho}{2\beta}, \quad (4)$$

$$\frac{\dot{R}^2}{R^2} + \frac{\ddot{R}}{R} + \frac{k}{R^2} = (-)\frac{p}{2\beta}. \quad (5)$$

Here  $\beta = 8\pi$  (We use geometrized units),  $\rho = \rho(t)$  means the rest density of the fluid, and  $p = p(\rho) = p(t)$  — its pressure.  $\dot{R} := \frac{dR}{dt}$ , and  $\ddot{R} := \frac{d^2R}{dt^2}$ .

Caloric equation  $p = p(\rho)$  must be added to Friedman equations (4)-(5) in order to get a determined system on the three unknown functions:  $R = R(t)$ ,  $\rho = \rho(t)$ ,  $p = p(t)$ .

Usually one considers solutions to the Friedman equations (4)-(5) in the two extreme cases:  $p = 0$  (dust universes or matter dominant universes, in short **MDU**), and  $p = \frac{\rho}{3}$  (radiation dominant universes, in short **RDU**).

We will confine to solutions in these two extreme cases.

Dust universes (**MDU**) with  $p = 0$ :

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<sup>3</sup>A particle of this fluid represents a cluster of galaxies in real Universe.

1.  $k=1$  (closed universe). In this case we have parametric solution

$$\begin{aligned} R &= M(1 - \cos \eta), \\ t &= M(\eta - \sin \eta). \end{aligned} \quad (6)$$

$$0 < \eta < 2\pi.$$

2.  $k=0$  (flat universe). In this case

$$R = \left(\frac{9M}{2}t^2\right)^{1/3}, \quad 0 < t < \infty. \quad (7)$$

3.  $k=(-)1$  (open universe). In the case we also have parametric solution

$$\begin{aligned} R &= M(\cosh \eta - 1), \\ t &= M(\sinh \eta - \eta), \quad 0 < \eta < \infty. \end{aligned} \quad (8)$$

Here  $\eta$  denotes a parameter and  $M = (4/3)\pi R^3 \rho$  is the first integral of the Friedman equations. Physically  $M$  is the mass contained inside of a “sphere” having volume  $(4/3)\pi R^3$ .

Radiation universes (**RDU**) with  $p = \frac{\rho}{3}$

1.  $k=1$  (closed universe)

$$R = \sqrt{(2bt - t^2)}, \quad b := \sqrt{\frac{8\pi C}{3}}, \quad 0 < t < 2b, \quad (9)$$

where  $C = \rho R^4 = \text{const} > 0$  is the first integral of the Friedman equations in this case.

2.  $k=0$  (flat universe)

$$R = \sqrt{2bt}, \quad 0 < t < \infty. \quad (10)$$

3.  $k=(-)1$  (open universe)

$$R = \sqrt{(2bt + t^2)}, \quad 0 < t < \infty. \quad (11)$$

Having  $R = R(t)$  one can find  $\rho(t)$  from the first integrals and then  $p = p(t)$  from caloric equations.

It is believed that one of the **MDU** correctly describes present stage of the Universe, and that one of the **RDU** correctly describes early Universe<sup>4</sup>.

It is seen from (6)-(11) that the Friedman universes are singular at least in one moment of the cosmic time  $t$  (In this moment  $R = 0$ ). These singularities are inevitable in classical general relativity (Theorems by Hawking and Penrose, and Senovilla [3]); but “quantized general relativity” (loops quantum gravity) seems remove these singularities (Ashtekar, Bojowald and Lewandowski) [4].

## 2 Conformal rescaling of metric and conformally flat spacetimes

By *conformal rescaling* of the metric  $g$  we mean the following transformation (in established coordinates)

$$\hat{g}_{ab}(x) = \Omega^2(x)g_{ab}(x), \quad (12)$$

where the *conformal factor*  $\Omega(x)$  is dimensionless, smooth and positive.

One can immediately get from (12) that

$$\hat{g}^{ab}(x) = \Omega^{(-)2}(x)g^{ab}(x), \quad (13)$$

and, after some tedious calculations one can obtain other useful transformational formulas [5]. For our future aims the following formulas will be needed

$$\begin{aligned} \hat{R}^b{}_d &= \Omega^{(-)2}R^b{}_d + 2\Omega^{(-)1}(\Omega^{(-)1})_{;dc}g^{bc} \\ &\quad - \frac{1}{2}\Omega^{(-)4}(\Omega^2)_{;ac}g^{ac}\delta^b_d, \\ \hat{R} &= \Omega^{(-)2}R - 6\Omega^{(-)3}\Omega_{;cd}g^{cd}, \end{aligned} \quad (14)$$

and

$$\hat{T}_i{}^k = \Omega^{(-)4}T_i{}^k. \quad (15)$$

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<sup>4</sup>The recent large-scale astronomical observations seem favorize an accelerated flat model.

Here ; $a$  is covariant derivative with respect Levi-Civita connection of the metric in the initial gauge  $g_{ab}(x)$ .

A spacetime is *conformally flat* if there exist holonomic coordinates ( $x^0 = t$ ,  $x^1 = x$ ,  $x^2 = y$ ,  $x^3 = z$ ) in which its line element  $ds^2$  has the form

$$\begin{aligned} ds^2 &= \Omega^2(x^0, x^1, x^2, x^3)(dx^{0^2} - dx^{1^2} - dx^{2^2} - dx^{3^2}) \\ &\equiv \Omega^2(x^0, x^1, x^2, x^3)\eta_{ik}dx^i dx^k. \end{aligned} \quad (16)$$

The

$$\eta_{ik}dx^i dx^k = dx^{0^2} - dx^{1^2} - dx^{2^2} - dx^{3^2} \quad (17)$$

means the line element of the empty, flat Minkowski spacetime in inertial coordinates.

The Theorem is true:

Necessary and sufficient condition of the conformal flateness of the 4-dimensional (or higher,  $n > 4$  dimensional) spacetime is vanishing its *Weyl tensor*  $C_{abcd}$ , where

$$\begin{aligned} C_{abcd} &:= R_{abcd} + \frac{R(g_{ac}g_{bd} - g_{ad}g_{bc})}{(n-1)(n-2)} - \\ &- \frac{(g_{ac}R_{bd} - g_{bc}R_{ad} + g_{bd}R_{ac} - g_{ad}R_{bc})}{(n-2)}. \end{aligned} \quad (18)$$

In the above formula  $R_{abcd}$  are components of the Riemann tensor,  $R_{ab}$  denote Ricci tensor components and  $R$  means Riemannian curvature scalar.

In the framework of general relativity Weyl's tensor  $C_{abcd}$  describes free gravitational field (tidal forces).

An example of the conformally flat spacetimes give Friedman universes.

### 3 Conformal transformation as Creator of the Friedman universes

We have under conformal rescaling of the metric (12) if we use the formulas (14)-(15)

$$\begin{aligned} \hat{G}_b^d &= \hat{R}_b^d - \frac{1}{2}\delta_b^d \hat{R} = \Omega^{(-)2} G_b^d \\ &+ \frac{2}{\Omega}(\Omega^{(-)1})_{;bc} g^{dc} + \frac{3}{\Omega^3} \delta_b^d \Omega_{;ce} g^{ce} \\ &- \frac{1}{2\Omega^4}(\Omega^2)_{;ac} g^{ac} \delta_b^d; \end{aligned} \quad (19)$$

$$\hat{T}_b^d = \Omega^{(-)4} T_b^d. \quad (20)$$

By using Einstein equations in *old gauge*  $g_{ik}(x)$

$$G_b^d = \beta T_b^d \quad (21)$$

one can combine (19)-(20) to the form

$$\hat{G}_b^d = \beta \Omega^2 \hat{T}_b^d + \beta \tilde{T}_b^d, \quad (22)$$

where

$$\begin{aligned} \tilde{T}_b^d &:= \frac{1}{\beta} \left[ \frac{2}{\Omega} (\Omega^{(-)1})_{;bc} g^{dc} \right. \\ &\quad \left. + \frac{\delta_b^d}{\Omega^3} (3\Omega_{;ce} g^{ce} - \frac{\Omega^2_{;ac}}{2\Omega} g^{ac}) \right]. \end{aligned} \quad (23)$$

(22) gives Einstein equations in *new gauge*  $\hat{g}_{ik}(x)$ .

The tensor  $\tilde{T}_b^d(x)$  is the energy-momentum tensor of this matter *which was created* by conformal rescaling of the initial metric  $g_{ik}(x)$  while the tensor  $\hat{T}_b^d(x)$  is transformed, following (20), the matter tensor  $T_b^d(x)$  which have already existed in the *old gauge*  $g_{ik}(x)$ .

One can rewrite (22) to the form

$$\hat{G}_b^d = \beta \bar{T}_b^d, \quad (24)$$

where

$$\bar{T}_b^d := \Omega^2 \hat{T}_b^d + \tilde{T}_b^d. \quad (25)$$

Of course, the *total matter* tensor (25) is covariantly conserved.

Friedman universes are conformally flat. So, we can take in the case as “initial conditions”

$$g_{ik}(x) = \eta_{ik}, \quad G_b^d = 0, \quad T_b^d = 0 \longrightarrow \hat{T}_b^d(x) = 0, \quad (26)$$

i.e., we can take empty Minkowskian spacetime as initial spacetime. Doing so, one can get the metric tensor of a Friedman universe in the form

$$\hat{g}_{ik}(x) = \Omega^2(x) \eta_{ik}, \quad (27)$$

where conformal factor  $\Omega(x)$  depends on Friedman universe.

Thus, metric  $\hat{g}_{ik}(x)$  of a Friedman universe, i.e., *whole geometry* of a Friedman universe can be obtained from empty Minkowskian spacetime by a suitable conformal rescaling of the Minkowskian metric. Material content of this universe can be easily obtained from Einstein equations

$$\tilde{T}_b^d := \frac{1}{\beta} \hat{G}_b^d, \quad (28)$$

where  $\hat{G}_b^d(x)$  is Einstein tensor calculated from  $\hat{g}_{ik}(x)$  or, immediately, from equations (23).

As an example we will consider a flat Friedman universe.

In this case

$$\hat{g}_{ik}(x) = \Omega^2(\tau) \eta_{ik} = \Omega^2(\tau) (d\tau^2 - dx^2 - dy^2 - dz^2) \quad (29)$$

with  $\Omega(\tau) \equiv R(\tau)$ .  $\tau$  is here the so-called *conformal time* [6].

After a simple but tedious calculation one gets from (28) [or from (23)] that

$$\begin{aligned} \tilde{T}_0^0 &= \frac{3R'}{\beta R^4} \quad (= \rho) \\ \tilde{T}_1^1 &= \tilde{T}_2^2 = \tilde{T}_3^3 = \frac{1}{\beta R^3} (2R'' - \frac{R'^2}{R}) \quad (= -p). \end{aligned} \quad (30)$$

Here prime denotes derivation with respect conformal time  $\tau$ .

Other components of the energy-momentum tensor  $\tilde{T}_b^a$  of the matter created by conformal rescaling (29) of the Minkowskian metric are vanishing.

For the flat dust Friedman universe we obtain

$$ds^2 = R^2(\tau) (d\tau^2 - dx^2 - dy^2 - dz^2), \quad (31)$$

where

$$R(\tau) = \frac{A^3}{9} \tau^2, \quad A = (6\pi\rho R^3)^{1/3} = \text{const}. \quad (32)$$

From that one gets

$$R' = \frac{2A^3\tau}{9}, \quad R'' = \frac{2A^3}{9}, \quad R''' = 0, \quad (33)$$

and higher derivatives also vanish.



In consequence, the material content of the universe following (30) reads

$$\tilde{T}_0^0 = \rho = \frac{972}{\beta A^6 \tau^6}. \quad (34)$$

The other components of the tensor  $\tilde{T}_a^b$  are vanishing, i.e.,  $p = 0$  and stresses vanish (as it should be in the case).

Thus, we have correctly created flat, dust Friedman universe from empty Minkowskian spacetime by using the conformal transformation (31)-(32).

## 4 Conclusion

As we could see, Friedman universes *can be created* by a suitable conformal rescaling of the flat Minkowskian metric, i.e., these universes *can be created* from empty, flat Minkowskian spacetime by conformal transformations.

Therefore, we needn't any "quantum gravity" in order to explain origin of the Friedman universes: classical conformal transformations are sufficient.

The analogical statement is, of course, correct for any other conformally flat spacetime.

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## **Interesująca własność modeli kosmologicznych Friedmana**

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### **Streszczenie**

W tej pracy pokazano, że modele kosmologiczne Friedmana, które są podstawą współczesnej kosmologii, można wykreować z pustej czasoprzestrzeni Minkowskiego przy pomocy odpowiedniej transformacji konforemnej.